Free Oscillations of a Linear Oscillator

Manual

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Annotation. The manual includes a description of the simulated physical system and a summary of the relevant theoretical material for students as a prerequisite for the virtual lab "Free Oscillations of Linear Torsion Pendulum." The manual includes also a set of theoretical and experimental problems to be solved by students on their own, as well as various assignments which the instructor can offer students for possible individual mini-research projects.

Contents

1 Summary of the Theory

1.1 General Concepts

Equilibrium of a physical system is called *stable* if under any disturbance a *restoring force* (or a restoring torque) arises which tends to return the system to the equilibrium position. If the system is disturbed from this state by some external initial action and then left to itself, the system subsequently oscillates about its equilibrium position. Such unforced oscillations are called *free* or *natural,* because they are not driven by an external force. Any system that is able to execute free oscillations in the vicinity of its position of stable equilibrium is called an *oscillator*.

In the absence of friction, energy transmitted to the system at the initial excitation remains constant during subsequent oscillations. Therefore, in such idealized *conservative* system with one degree of freedom, the motion is strictly *periodic* and continues indefinitely without damping. If there is friction in the system, free oscillations are *damped*: they gradually fade away because of the dissipation of energy, and the system eventually comes to rest in the equilibrium position.

If the restoring force which tends to return a disturbed conservative system to its equilibrium position is proportional to the displacement from this position, oscillations of the system are *harmonic*. When there is no friction, or when the damping force is proportional to the velocity and oppositely directed (viscous friction), the differential equation describing the motion of the system is *linear* because the displacement and its time derivatives are to the first power.

1.2 Differential Equation of a Linear Torsion Oscillator

The linear oscillator simulated in this computer program is a balanced flywheel whose center of mass lies on the axis of rotation. Such a flywheel may consist, for example, of a rod with two equal masses, as shown in figure 1.

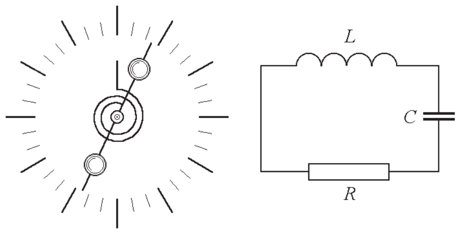

Figure 1: Schematic image of the torsion spring oscillator simulated in the program and its electromagnetic analog – LCR-circuit.

The rod can rotate about an axis that passes through its center. A spiral spring with one end fixed and the other attached to the flywheel creates a restoring torque N which is proportional to the angular displacement φ:

$$N = -D\varphi. \tag{1}$$

Here D is a constant of proportionality called the torsion *spring constant*. Measured in units of torque per radian, its value depends on the strength of the spring. The angular displacement of the flywheel from its equilibrium position is measured by a needle attached to the flywheel, and a fixed dial.

The law of rotation of a flywheel whose moment of inertia about its axis of symmetry is J, and which is acted upon by a torque N in the absence of friction, gives the following differential equation:

$$J\ddot{\varphi} = -D\varphi \qquad \text{or} \qquad \ddot{\varphi} + \omega_0^2\varphi = 0, \tag{2}$$

where $\omega_0^2 = D/J$. The general solution of Eq. (2) yields a simple harmonic oscillation:

$$\varphi(t) = A_0 \cos(\omega_0 t + \delta_0), \tag{3}$$

whose amplitude A_0 and initial phase δ_0 depend on the *initial conditions* (that is, on the angular displacement $\varphi(0)$ and the angular velocity $\dot{\varphi}(0)$ at $t = 0$). In other words, these parameters of the motion depend on the way in which oscillations are excited.

The oscillations occur with the frequency ω_0, the squared value of which is proportional to the spring constant D and inversely proportional to the moment of inertia J of the flywheel. The frequency ω_0 and the corresponding period $T_0 = 2\pi/\omega_0$, unlike the amplitude and initial phase, do not depend on the initial conditions—they are entirely determined by the properties of the system, i.e., by the values of the parameters D and J. Free oscillations of the system always occur with the same *natural frequency* ω_0, independently of the mode of excitation.

When the flywheel is also acted upon by a force of viscous friction which is proportional to and oppositely directed to the angular velocity $\dot{\varphi}$, the differential equation of motion has the form:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2\varphi = 0, \tag{4}$$

where the *decay* or *damping constant* γ characterizes the strength of viscous friction in the system.

Equations (2) and (4) are *linear* differential equations because the dependent variable, $\varphi(t)$, and its time derivatives occur only to the first power. We shall therefore refer to this system as a *linear oscillator*. These equations are also *homogeneous*, because $\varphi(t)$, or its derivatives, appear to the same power in every term of the equation. The homogeneity of Eqs. (2) and (4) implies that an external driving force, independent of $\varphi(t)$ or its time derivatives, is not present. We call the oscillations described by Eqs. (2) and (4) as *free* or *natural* oscillations. They occur when there is no external driving force.

As for any homogeneous equation, Eqs. (2) and (4) have the trivial solution, $\varphi(t) = 0$ and $\dot{\varphi}(t) = 0$. This solution describes a system that is always at rest in its equilibrium position. Since the solutions to this second-order differential equation are completely determined by the values of φ and $\dot{\varphi}$ at some particular moment, it is clear that if the initial values of φ and $\dot{\varphi}$ are zero, they must remain zero forever. Hence, if non-zero values φ and $\dot{\varphi}$ are to be found, $\varphi(0)$ and $\dot{\varphi}(0)$ cannot both be zero. Oscillations of the system are produced only when there is some initial excitation of the system.

When friction is so weak that $\gamma < \omega_0$, the general solution of Eq. (4) can be written in the form:

$$\varphi(t) = A_0 \exp(-\gamma t) \cos(\omega_1 t + \delta_0). \tag{5}$$

This solution describes *damped oscillations* whose amplitude $A_0 \exp(-\gamma t)$ decreases exponentially with time. The amplitude constant A_0 and the initial phase δ_0 depend on the initial conditions. The frequency ω_1 appearing in the cosine term in (5) is given by

$$\omega_1 = \sqrt{\omega_0^2 - \gamma^2} = \omega_0 \sqrt{1 - (\gamma/\omega_0)^2}. \tag{6}$$

In the case of relatively weak damping, when constant γ is small compared to the natural frequency ω_0, frequency ω_1 is very close to ω_0:

$$\omega_1 \approx \omega_0 - \gamma^2/(2\omega_0).$$

The fractional difference $(\omega_0 - \omega_1)/\omega_0$ of these frequencies is proportional to the square of the small parameter γ/ω_0.

Graphs of the deflection angle and of the angular velocity for oscillations damped by the force of viscous friction are shown in Fig. 2.

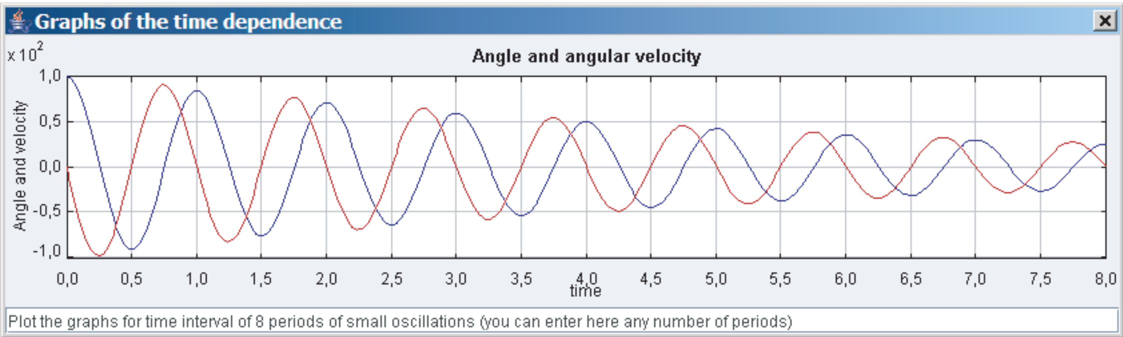

Figure 2: Plots of the deflection angle and of the angular velocity.

1.3 The Time of Damping and the Quality Factor Q

In the case of moderate damping, the time-dependent factor $A_0 \exp(-\gamma t)$ in (5) can be treated as a slowly decreasing amplitude of diminishing oscillations. After an interval $\tau = 1/\gamma$, the amplitude is $e \approx 2.72$ times smaller than its initial value. The time τ is called the *decay time* or the *time of damping*.

When $\gamma \ll \omega_0$, or $\tau \gg T_0 = 2\pi/\omega_0$ (condition of *weak damping*), the oscillator executes a large number N of oscillations during the decay time τ: $N = \tau/T_0 \gg 1$. Consecutive maximal deflections from the equilibrium position diminish in a geometric progression. Letting φ_n be the maximal angular displacement of the n-th oscillation, we have

$$\varphi_{n+1}/\varphi_n \approx \exp(-\gamma T_0) \approx 1 - \gamma T_0.$$

That is, the ratio of successive terms in this infinite geometric progression is less than unity by the small value $\gamma T_0 = T_0/\tau \ll 1$.

The strength of viscous friction in the system is usually characterized either by the damping constant γ, which, as can be seen from Eq. (4), has the dimension of frequency, or by a dimensionless quantity Q, called the *quality factor.* The quality factor is defined by:

$$Q = \frac{\omega_0}{2\gamma} = \pi \frac{\tau}{T_0}. \tag{7}$$

The number of cycles during which the amplitude of oscillations decreases by a factor $e \approx 2.72$ is given by Q/π, and the number of cycles $N_{1/2}$ during which the amplitude is halved is given by:

$$N_{1/2} = (\ln 2/\pi)Q = 0.22\,Q = Q/4.53. \tag{8}$$

When $\gamma \geq \omega_0$ (condition of *strong damping*), a disturbed oscillator returns to the equilibrium position without oscillating. In this motion, the oscillator either approaches the equilibrium position asymptotically from one side or overshoots the equilibrium position once and then asymptotically reapproaches it from the other side. This latter case occurs only when the initial angular velocity of the oscillator is directed toward the equilibrium position, and its magnitude is large enough.

When $\gamma = \omega_0$, the system is said to be *critically damped.* The general solution to the differential equation of motion, Eq. (4), for the critically damped system takes the form:

$$\varphi(t) = (C_1 t + C_2)\exp(-\gamma t), \tag{9}$$

where C_1 and C_2 are constants defined by the initial conditions. For example, if the system is given an initial velocity at the equilibrium position, that is, if $\varphi(0) = 0$ and $\dot{\varphi}(0) = \Omega_0$, then $C_1 = \Omega_0$, $C_2 = 0$, and the motion of the system is described by the function:

$$\varphi(t) = \Omega_0 t \exp(-\gamma t). \tag{10}$$

An interesting feature of the critically damped system is that, after an initial disturbance, it returns to rest in the equilibrium position usually sooner than it does in any other case (i.e., than it does for any other value of the damping constant γ for a given value of ω_0). It is seen from Eq. (7) that the value of the quality factor which corresponds to critical damping ($\gamma = \omega_0$) is $Q = 0.5$.

Non-oscillatory motion at strong friction, when $\gamma > \omega_0$, can be represented as a superposition of two exponential functions, which have different time constants $\tau_1 = -1/\alpha_1$ and $\tau_2 = -1/\alpha_2$:

$$\varphi(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t}, \qquad \text{where } \alpha_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}. \tag{11}$$

Values of C_1 and C_2 are determined by the initial conditions.

In measuring instruments such as moving-coil galvanometers, damping is introduced intentionally in order to overcome the difficulty of taking a reading from an oscillating needle. A measuring instrument is said to be *critically damped* if the needle just fails to oscillate and comes to rest in the shortest possible time.

If the instrument is *underdamped* ($Q > 0.5$), the needle oscillates repeatedly before coming to rest. If the instrument is *overdamped* ($Q < 0.5$), the needle does not oscillate, though it takes longer to come to rest than it does when the instrument is critically damped.

1.4 The Phase Diagram of a Linear Oscillator

The mechanical state of a torsion pendulum at any instant is determined by the two quantities: the *angular displacement* φ and the *angular velocity* $\dot{\varphi}$ (or instead by the *angular momentum* $J\dot{\varphi}$). The evolution of the mechanical state of the system during its entire motion can be graphically demonstrated very clearly in a *phase diagram.* This is a graph which plots the angular velocity $\dot{\varphi}$ (or the angular momentum $J\dot{\varphi}$) versus the angular displacement φ.

The mechanical state of the system at any instant is represented by a point, called the *representative point*, in the phase plane. If the motion of the physical system is *periodic*, the representative point, moving clockwise, generates a *closed path* in the phase plane. The phase trajectory of periodic motion is closed, because the system returns to the same mechanical state after a full cycle.

The phase diagram for harmonic oscillations (e.g., for oscillations in a linear system without friction) is an ellipse (or a circle at the appropriate choice of the scales). The points of intersection of the phase curve with the φ-axis (the *extreme,* or *turning points*) correspond to maximal deflections of the oscillator from the equilibrium position. At these points, the sign of the angular velocity $\dot{\varphi}$ changes, and the tangent to the phase curve is perpendicular to the abscissa axis.

As noted above, the period of harmonic oscillations is determined entirely by the parameters of the physical system, specifically by the values of the spring constant D and the moment of inertia J. Unlike the amplitude and the initial phase, the period does not depend on the initial conditions, that is, on the way oscillations are excited. This property of the harmonic oscillator is called *isochronism.* Because of this property, the representative point generates ellipses of different sizes (which correspond to various amplitudes of oscillations in the same system) during the same time T_0.

In the presence of relatively weak viscous friction ($\gamma < \omega_0$) the extreme displacements, as well as the extreme values of the angular velocity, gradually diminish with each subsequent cycle of oscillation. Consequently the phase trajectory for free oscillations is transformed from a closed curve (an ellipse or a circle) into a shrinking spiral that winds around a focal point located at the origin of the phase plane. Figure 3 shows the phase diagram of damped oscillations together with the parabolic potential well. During the oscillations, the representing points travels in this well from one slope to the other descending gradually to the bottom of the well.

The family of phase trajectories, which correspond to the same values of the parameters of the system but to different initial conditions forms a *phase portrait* of the system. This portrait gives a clear graphic representation of all possible motions of the system.

The phase portrait of a conservative linear oscillator is formed by a set of similar ellipses with a common *center* at the origin of the phase plane. The center represents a state of rest in the equilibrium position. When friction is relatively weak ($\gamma < \omega_0$), this center becomes an attractor of the phase

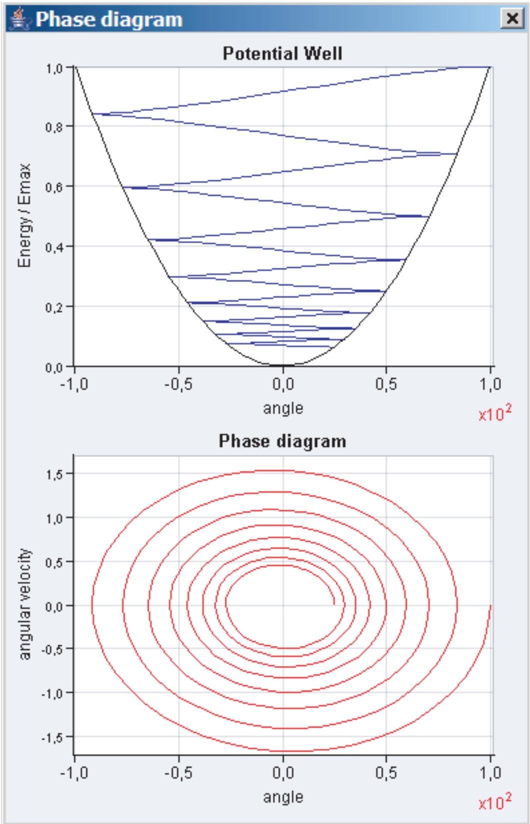

Figure 3: The parabolic potential well and the phase diagram of damped oscillations.

trajectories called the *focal point*. That is, all phase trajectories of damped oscillations spiral in toward the origin, forming an infinite number of gradually shrinking loops, as in Fig. 4,*a*.

When friction is relatively strong ($\gamma > \omega_0$), the attractor of the phase trajectories becomes a *node*: all phase trajectories of non-oscillatory motion approach this node directly, without spiraling. The phase portrait of an overdamped system ($\gamma = 1.05\,\omega_0$) is shown in Fig. 4,*b*. The phase curves asymptotically approach the origin, where they have a common tangent $\dot{\varphi} = \alpha_1\varphi$, where $\alpha_1 = -\gamma + \sqrt{\gamma^2 - \omega_0^2}$. At specific initial conditions, when $\dot{\varphi}(0) = \alpha_1\varphi(0)$, the representative point moves towards the node directly along this tangent. The other rectilinear phase trajectory $\dot{\varphi} = \alpha_2\varphi$ ($\alpha_2 = -\gamma - \sqrt{\gamma^2 - \omega_0^2}$) occurs at initial conditions of the kind $\dot{\varphi}(0) = \alpha_2\varphi(0)$.

In general, only one phase trajectory passes through a given point of the phase plane. Indeed, if we consider this arbitrary point as an initial state of the system, the further motion of the system is defined uniquely. This motion is represented by a single phase trajectory, passing through the point.

However, there may be exceptions in that either no phase trajectory passes through a phase point, or there are several trajectories at once. Phase points of this kind are called *singular*.

For a linear oscillator, there is only one singular point: the origin of the phase plane. It corresponds

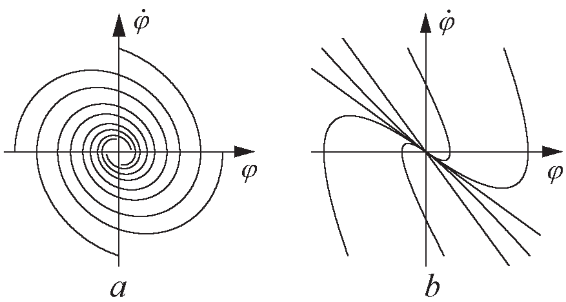

Figure 4: Phase portrait of an underdamped (*a*) and of an overdamped (*b*) linear oscillator with viscous friction.

to the state of rest in the equilibrium position, where both φ and $\dot\varphi$ are zero. When $\gamma = 0$, this point is a center, and no phase trajectory passes through it. When $0 < \gamma < \omega_0$, or when $\gamma > \omega_0$, it is respectively a focus or a node, to which all phase trajectories are attracted. In the lab "Oscillations and rotations of a rigid pendulum," we encounter another kind of singular point in a conservative system, the saddle point.

1.5 Energy Transformations

The total energy E of the torsion spring pendulum is the sum of the elastic *potential energy* $E_{\rm pot}$ of the strained spring and the *kinetic energy* $E_{\rm kin}$ of the rotating flywheel:

$$E = E_{\rm pot} + E_{\rm kin} = \frac{1}{2}D\varphi^2 + \frac{1}{2}J\dot\varphi^2. \tag{12}$$

Free oscillations in the absence of friction are characterized by the *exchange of energy* between its potential and kinetic forms. At the points of maximum displacement from the equilibrium position, the kinetic energy is zero, and the total energy of the oscillator is the potential energy of the strained spring. A quarter of a period later, the oscillator passes through the equilibrium point, where the potential energy is zero and the total energy of the oscillator is the kinetic energy of the flywheel. During the next quarter of a period, the reverse exchange of energy occurs: the kinetic energy is transformed into potential energy. Such transformations happen twice during one period. That is, oscillations of the two kinds of energy, 180° out of phase with one another, are executed with the frequency $2\omega_0$, i.e., with double the natural frequency ω_0 of the system.

The exchanges between potential and kinetic energy described above are characteristic of a conservative system, in which such transformations are reversible. The sum of the kinetic and potential energy, i.e., the total mechanical energy E of the oscillator, is the same at every instant and is equal to the maximum values of both kinetic energy and potential energy:

$$E = \frac{1}{2}DA_0^2 = \frac{1}{2}J\omega_0^2A_0^2. \tag{13}$$

The total energy of the system is proportional to the square of the amplitude A_0. The values of the two forms of energy, averaged over a period, are each equal to one half of the total energy:

$$\langle E_{\rm pot}\rangle = \langle E_{\rm kin}\rangle = \frac{1}{2}E = \frac{1}{4}DA_0^2 = \frac{1}{4}J\omega_0^2A_0^2. \tag{14}$$

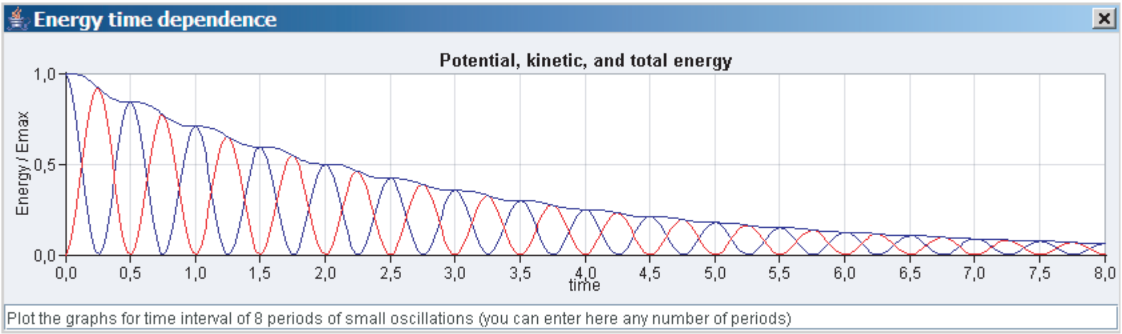

Figure 5: Energy transformations during damped natural oscillations

In the presence of friction, the exchanges between kinetic and potential energy are partially irreversible because of the dissipation of mechanical energy. This dissipation occurs nonuniformly during a complete cycle: its rate, $-dE/dt$, is zero when the flywheel, in a given cycle, is at the extremes of its motion and its angular velocity, $\dot{\varphi}$, is zero. This is clearly seen in figure 5 which shows the transformations of energy occurring in damped oscillations. The rate of dissipation is greatest when the flywheel moves in the vicinity of the equilibrium position, where its angular velocity is maximal. Indeed, the rate of energy dissipation $-dE/dt = -N_{\rm fr}\dot{\varphi}$ caused by viscous friction is proportional to the square of the angular velocity and hence to the momentary value of the kinetic energy $J\dot{\varphi}^2/2$ of the oscillator. The statement, frequently encountered in textbooks, that the energy of a damped oscillator decays exponentially, applies only to values of the total energy averaged over a period.

1.6 The Computer Simulation of Linear Oscillator

The computer simulation of the torsion spring pendulum is based on the numerical integration of the differential equation, Eq. (4). Although this equation can be integrated analytically, the analytic solution is not used in those parts of the computer program which demonstrate the evolution of the system in time. It is used only to determine the maximal values of the angular displacement and of the angular velocity in order to establish suitable scales for the corresponding plots. The agreement between the results of the numerical integration in the computer simulation and the analytic predictions can serve as a confirmation of the quality of the algorithm used (the Runge—Kutta method taken to fourth order). This verification gives us confidence in the reliability of the the computer simulations of nonlinear systems in subsequent programs of this software package since the simulations are based on the same numerical method.

In this numerical simulation of the linear oscillator we adopt a unit of time which is appropriate for the system under consideration, namely the period $T_0 = 2\pi/\omega_0$ of free oscillations in the absence of friction. Thus the simulated oscillator may be characterized by only one physical parameter: either by the dimensionless ratio of the damping constant to the natural frequency γ/ω_0, or by the equivalent dimensionless quantity—the quality factor $Q = \omega_0/(2\gamma)$, inversely proportional to γ/ω_0.

The angular displacement, φ, is expressed in radians in the program, though for convenience of observation, the dial on the screen and the plots involving the angle of deflection are graduated in degrees. The angular velocity, $\dot{\varphi}$, is measured in units of the natural frequency, ω_0. When the initial conditions are set in a simulation computer experiment, the initial angular velocity also must be expressed in units of ω_0.

2 Questions, Problems, Suggestions

2.1 Free Undamped Oscillations

1.1 **The Initial Conditions and the Shape of the Plots.**

In the absence of friction a linear oscillator executes simple harmonic motion, which is characterized by purely sinusoidal time dependence of the angular displacement and of the angular velocity.

(a) What initial conditions give rise to oscillations of cosine time dependence, of sine time dependence? Suppose that you want to get oscillations with the angular amplitude of $90°$. What initial angular displacement $\varphi(0) = \varphi_0$ at zero initial angular velocity $\dot{\varphi}(0) = 0$ ensures the desired amplitude?

(b) What initial angular velocity $\dot{\varphi}(0) = \Omega$ ought you to impart to the oscillator, at rest in the equilibrium position, in order to obtain the same amplitude of $90°$? Remember, that the initial angular velocity Ω must be expressed for input in units of the natural frequency ω_0. Verify your answer with a computer experiment, using the appropriate initial conditions.

1.2 **Maximal Deflection and Conservation of Energy.** Imagine exciting an oscillator initially at rest in the equilibrium position by a push which produces an initial angular velocity $\Omega = 2\omega_0$.

(a) Calculate the angle $\varphi_{\max}$ of maximal deflection using the law of the conservation of energy.

(b) Verify your result experimentally. Note that the simulation program performs the numerical integration of the differential equation independently of conservation laws, such as the conservation of energy. That is, these laws are not used in the program.

1.3 **The Phase Trajectory and the Initial Conditions.** Compare the motion of the representative point along the phase trajectory of a conservative oscillator with the time-dependent plots of the angle of deflection and of the angular velocity.

(a) How is the phase trajectory changed if you change the initial conditions?

(b) Does the direction of the motion of the representative point along the phase trajectory depend on the initial conditions?

(c) Is it possible that phase trajectories for different initial conditions coincide? If so, formulate the requirements for the coincidence.

1.4 **Elliptical and Circular Shape of the Phase Trajectory.**

(a) Prove analytically that the phase trajectory of a conservative linear oscillator is an ellipse with its center at the origin of the phase plane. Use the general solution of Eq. (2), expressed by Eq. (3). What are the semiaxes of the ellipse?

(b) Show that the elliptical shape of the phase diagram of a conservative linear oscillator follows immediately from the law of the conservation of the energy.

(c) What scale on the axis of the ordinate (the angular velocity axis) of the phase plane produces a circular phase trajectory?

(d) Does the time interval during which the representative point passes along one loop of the phase trajectory depend on the initial conditions?

1.5 **The Phase Diagram and Energy Transformations.** Compare the phase trajectory with the plot of potential energy versus the angle of deflection. The positioning of plots on the display screen (if you open the window "Phase diagram") is convenient for such comparison. Pay special attention to the positions of the extreme points on the phase trajectory and in the parabolic potential well. For the initial conditions $\varphi(0) = \varphi_0$, $\dot{\varphi}(0) = \Omega$, what are the values of the potential energy and the kinetic energy at the extreme points and at the equilibrium position?

What are the extreme deflection $\varphi_{\max}$ and the maximal angular velocity $\omega_{\max}$ of the flywheel?

1.6 **The Shape and the Frequency of Energy Oscillations.** Consider the plots of the time dependence of kinetic energy and potential energy.

(a) What can you say about their maximal and average values? Compare these plots with the plots of the angular displacement and the angular velocity.

(b) At what frequency do the oscillations of each kind of energy occur? What are the limits (the extreme values) and the mean (averaged over a period) values of each kind of energy in these oscillations?

1.7 **The Phase Trajectories with the Same Energy.** Consider the oscillations of a conservative oscillator at different initial conditions but with the same total energy. What differences do you observe in the plots and the phase trajectories in these cases?

2.2 Damped Free Oscillations

2.1 **The Sequence of Maximal Deflections.** Under the action of a weak force of viscous friction, the sequence of maximal deflections of a free, damped linear oscillator forms a decreasing geometric progression: each consecutive maximal deflection is smaller than the preceding one by the same factor, $\exp(-\gamma T_0) \approx 1 - \gamma T_0$ [see Eq. (7)].

(a) Calculate the value of the quality factor Q at which the amplitude halves during every two complete oscillations.

(b) Input this value in a computer experiment and verify the theoretically predicted constant ratio of successive maximal deflections. Note that this ratio does not depend on the initial conditions.

(c) Evaluate the increment of the period of oscillations at this value of the quality factor with respect to the period T_0 in the absence of friction (in percent). Can you detect the increment in the simulation experiment? The marks on the time axis correspond to integer numbers of periods $T_0 = 2\pi/\omega_0$ without friction.

2.2* **Maximal Deflection after an Initial Push.** Imagine, that we excite oscillations with an initial push which imparts an initial angular velocity of $2\omega_0$ to the flywheel in its equilibrium position.

(a) Calculate the first maximal deflection of the flywheel for the quality factor $Q = 5$.

(b) What will be the value of the subsequent extreme deflection which occurs in the direction opposite to the first? Verify your answers.

2.3** **Complex Initial Conditions.**

(a) Let the initial deflection of the torsion pendulum be 155 degrees, and the initial angular velocity be $2\omega_0$. The quality factor $Q = 5$. Calculate the maximal deflection of the flywheel.

(b) With the same initial deflection (155 degrees) and the same quality factor $Q = 5$ as in the preceding item (a), calculate the maximal deflection of the flywheel, if the initial angular velocity equals $-2\omega_0$.

(c) Let the initial deflection of the torsion pendulum be -155 degrees. What initial angular velocity would ensure the maximal deflection of 155 degrees (to the opposite side), if the quality factor $Q = 20$?

2.4* **The Phase Trajectory of Damped Oscillations.** The phase trajectory of damped free oscillations for $Q > 0.5$ is a spiral which makes an infinite number of gradually shrinking loops around the focus located at the origin of the phase plane. This focus corresponds to the state of rest in the equilibrium position, and the phase trajectory approaches it asymptotically.

(a) How does the radius of these loops change while the curve approaches the focus?

(b) Does the time interval during which the representative point makes one revolution of the spiral change as the loops of the curve shrink?

2.5* **The Dissipation of Energy.** Compare the transformation of potential energy into kinetic energy (and vice versa) for free undamped oscillations in the absence of friction with that for free damped oscillations in the presence of viscous friction.

(a) Show, using a simulation experiment, that if Q = 18.1, the amplitude is halved during four complete oscillations and the total energy is halved during two complete oscillations.

(b) Why is the dissipation of mechanical energy nonuniform during one cycle of oscillations? At what instants during a cycle is the time-rate of energy dissipation greatest and at what instants is it smallest?

2.3 Non-oscillatory Motion of the Pendulum

When viscous friction is strong ($Q \leq 0.5$), a disturbed system returns to the equilibrium position without oscillating. In the computer simulation, the needle asymptotically approaches the zero point from one side.

3.1* **Non-oscillatory Motion at Critical Damping.** Consider the case of critical damping, $\gamma = \omega_0$.

(a) Why is critical damping preferable in measuring instruments using a needle as an indicator? How might your answer apply to the suspension system in an automobile?

Show that the value of Q in the case of critical damping is 0.5.

(b) Calculate the maximal angle of deflection if the system, with $Q = 0.5$, receives an initial velocity $\Omega = 5\omega_0$ in the equilibrium position.

(c) In what lapse of time does the needle move towards this extreme point?

Verify your answers by simulating the experiment on the computer. Note that the needle approaches the equilibrium position from one side—it does not cross the zero point of the dial.

3.2 **Critical Damping.**

(a) Prove that the value $Q = 0.5$ ($\gamma = \omega_0$) is really critical. Do so by showing that at slightly greater values of Q, the needle of a perturbed oscillator executes heavily damped oscillations, slowly moving to and fro across the zero point of the dial. (Sound ticks at crossing the zero point may help you).

(b) For a critically damped system, express the constants C_1 and C_2 in the general solution $\varphi(t) = (C_1 t + C_2)\exp(-\gamma t)$ of the differential equation, Eq. (4), in terms of the initial displacement $\varphi(0) = \varphi_0$ and the initial angular velocity $\dot{\varphi}(0) = \Omega_0$.

(c) Is it possible for a critically damped system to move after an initial disturbance according to pure exponential law? If so, what initial conditions give rise to such motion? What is the phase trajectory of this motion? Prove your answers experimentally.

(d) At what initial conditions the flywheel of a disturbed critically damped system will cross the equilibrium position? For a given initial displacement φ_0, what initial angular velocity Ω should you impart to the flywheel of the critically damped oscillator in order it crossed the equilibrium position after a lapse of time $t = 3T_0$, where $T_0 = 2\pi/\omega_0$ is the natural period (the period of oscillations in the absence of friction)?

3.3* **Motion of an Overdamped System.**

(a) For arbitrary initial conditions ($\varphi(0) = \varphi_0$, $\dot{\varphi}(0) = \Omega_0$), express the values of C_1 and C_2 in the general solution (11) of the differential equation for an overdamped system in terms of φ_0 and Ω_0.

(b) At what initial conditions the motion of an overdamped system will be described by a monoexponential function of time? What are the phase trajectories that correspond to such motions?

(c) Explain, why at arbitrary initial conditions non-oscillatory motion of the flywheel towards the equilibrium position occurs more slowly and requires more time than at critical damping. Is it possible for an overdamped system to return to the equilibrium position faster than for the critically damped system with the same ω_0? If so, what conditions of excitation ensure the motion?

(d) What is the principal difference between the phase trajectories corresponding to a non-oscillatory motion and those corresponding to damped oscillations?

(e) Is it possible for an overdamped system ($\gamma > \omega_0$) to cross the equilibrium position after excitation? If so, what initial conditions give rise to such motion? Is it possible for the oscillator to cross the equilibrium position more than once?

2.4 Supplement: Review of the Principal Formulas

The differential equation of a free linear torsion oscillator:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2\varphi = 0.$$

The frequency and the period of free oscillations without friction (at $\gamma \ll \omega_0$):

$$\omega_0 = \sqrt{\frac{D}{J}}, \qquad T_0 = \frac{2\pi}{\omega_0}.$$

An oscillatory solution (valid at $\gamma < \omega_0$):

$$\varphi(t) = A_0 e^{-\gamma t}\cos(\omega_1 t + \delta_0),$$

where the constants A_0 and δ_0 are determined by the initial conditions $\varphi(0)$, $\dot{\varphi}(0)$. The frequency ω_1 of damped oscillations

$$\omega_1 = \sqrt{\omega_0^2 - \gamma^2}.$$

An equivalent form of the general solution:

$$\varphi(t) = e^{-\gamma t}(C\cos\omega_1 t + S\sin\omega_1 t),$$

where the constants C and S are determined by the initial conditions. They are related to A_0 and δ_0:

$$A_0 = \sqrt{C^2 + S^2}, \qquad \tan\delta_0 = -S/C.$$

In the case of weak damping ($\gamma \ll \omega_0$)

$$\omega_1 \approx \omega_0 - \gamma^2/(2\omega_0).$$

The decay time (during which the amplitude is reduced by the factor $e \approx 2.72$):

$$\tau = 1/\gamma.$$

A non-oscillatory motion at $\gamma = \omega_0$:

$$\varphi(t) = (C_1 t + C_2)e^{-\gamma t}.$$

The quality factor Q of an oscillator:

$$Q = \pi\frac{\tau}{T_0} = \frac{\omega_0}{2\gamma}.$$

The number of oscillations, during which the amplitude is halved:

$$N_{1/2} = \frac{\ln 2}{\pi}Q = 0.22\,Q = \frac{Q}{4.53}.$$

The total mechanical energy of the oscillator consists of elastic potential energy of the strained spring and kinetic energy of the flywheel:

$$E = E_{\mathrm{pot}} + E_{\mathrm{kin}} = \frac{1}{2}D\varphi^2 + \frac{1}{2}J\dot{\varphi}^2.$$

The values of the potential energy and kinetic energy of the oscillator, averaged over a cycle, equal one another, each of them constituting one half the total energy:

$$\langle E_{\mathrm{pot}}\rangle = \langle E_{\mathrm{kin}}\rangle = \frac{1}{2}E = \frac{1}{4}DA_0^2 = \frac{1}{4}J\omega_0^2A_0^2.$$