Oscillations of a simple pendulum with extremely large amplitudes

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Abstract. Large oscillations of a simple rigid pendulum with amplitudes close to 180° are treated on the basis of a physically justified approach in which the cycle of oscillation is divided into several stages. The major part of the almost closed circular path of the pendulum is approximated by the limiting motion, while the motion in the vicinity of the inverted position is described on the basis of the linearized equation. The accepted approach provides additional insight into the dynamics of nonlinear physical systems. The final simple analytical expression gives the values for the period of large oscillations that coincide with high precision with the values given by the exact formula.

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1. Introduction

The old problem of large oscillations of a simple planar pendulum continues to attract attention of the academic community. Dozens of papers on the subject appeared during the last decade in EJP, AJP, and other journals — see, for example, [1] – [5] and references therein. In most of the papers various approximation schemes have been developed to express the large-angle pendulum period by simple formulae in terms of elementary functions. Each of the authors usually claims that the formula proposed by him is more simple and accurate when compared with other approximate formulae. A detailed comparison of several approximate expressions that have appeared in recent publications can be found in [6]. The common feature of all suggested approximation schemes can be reduced to a search for some empirical expression for the period $T(\varphi_\mathrm{m})$ which gives for large amplitudes φ_m an acceptable numerical agreement with the values obtained from the exact formula given by the complete elliptic integral of the first kind $K(q)$:

$$T(\varphi_\mathrm{m}) = T_0\frac{2}{\pi}K(\sin^2(\varphi_\mathrm{m}/2)), \qquad K(q) = \int\limits_0^{\pi/2}\frac{dx}{\sqrt{1-q\sin^2 x}}. \tag{1}$$

where $T_0 = 2\pi/\omega_0 = 2\pi\sqrt{l/g}$ is the natural period and ω_0 is the frequency of oscillations with infinitely small amplitude, l is the effective length of the pendulum, and g is the acceleration due to gravity.

The approximate expressions for the period that can be found in the literature (see [6] and references therein) give indefinitely increasing errors as the amplitude of the pendulum tends to 180°. Moreover, all these exercises with various approximation schemes give little physical insight in the nonlinear dynamics of the pendulum behavior at large amplitudes.

In the present letter we suggest a radically different approach to the problem of extremely large amplitudes. Our approach is based on physically clear presentation of large oscillations as consisting of several stages during which the motion can be described analytically with high precision in terms of elementary functions. The principal idea of our approach is very simple: the motion of the pendulum in the close vicinity of the inverted position can be described by a linear differential equation (if we choose as a variable the angle $\alpha = \pi - \varphi$ which the pendulum makes with the upper vertical line), while the remaining part of the pendulum's path (constituting nearly a full circle) is almost indistinguishable from the limiting motion (motion along the separatrix), for which a simple solution in elementary functions is available. Precision of the final (very simple) formula for the period, equation (14), increases as the amplitude approaches 180°. We have already used this idea earlier in [7] while comparing large amplitude oscillations of the pendulum with full revolutions. The aim of the present letter is to draw attention to this fruitful approach that gives additional physical insight into the nonlinear dynamics of the pendulum – a very popular physical model often encountered in various undergraduate courses.

2. The phase portrait of the pendulum

Next we remind several peculiarities in the behavior of the simple pendulum which are essential for the problem of large-amplitude oscillations. The solution $\varphi(t)$ to the differential equation of a conservative simple pendulum

$$\ddot{\varphi} + \omega_0^2 \sin\varphi = 0 \qquad (\omega_0 = \sqrt{g/l}) \tag{2}$$

can be expressed in elementary functions in the limiting case of oscillations with infinitely small amplitude: when $\sin\varphi \approx \varphi$, equation (2) becomes linear and describes a simple harmonic motion with the frequency ω_0. Oscillations with large amplitudes, as well as revolutions in a full circle, require special functions (elliptic functions) for their description. However, the general character of variation with the time of the mechanical state of a nonlinear system such as the pendulum can be graphically demonstrated by trajectories in the phase plane $(\varphi, \dot{\varphi})$, i.e., the graphs which plot the angular velocity $\dot{\varphi}$ versus the angular displacement φ. The family of these trajectories, corresponding to different values of energy, constitutes the phase portrait of the system. The phase portrait tells us a great deal about the possible motions of a nonlinear system.

We can construct a phase portrait for a conservative system (e.g., for the pendulum) without explicitly solving the differential equation (2) of motion of the system. The equations for phase trajectories follow directly from the law of energy conservation. In the absence of friction, the total energy E of the pendulum, i.e., the sum of its kinetic energy, $E_{\mathrm{kin}}(\dot{\varphi}) = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\varphi}^2$, and potential energy, $E_{\mathrm{pot}}(\varphi) = mgl(1 - \cos\varphi)$, remains constant during the motion:

$$\frac{1}{2}ml^2\dot{\varphi}^2 + mgl(1 - \cos\varphi) = E. \tag{3}$$

This equation gives the relation between $\dot{\varphi}$ and φ, and therefore it is the equation of the phase trajectory which corresponds to a definite value E of total energy. It is convenient to express equation (3) in a slightly different form. We define the quantity $E_{\mathrm{m}} = 2ml^2\omega_0^2 = 2mgl$ that has the physical sense of the kinetic energy of the pendulum rotating with the angular velocity $2\omega_0$, or, which is the same, of the potential energy of the inverted pendulum. Then we rewrite equation (3):

$$\frac{\dot{\varphi}^2}{\omega_0^2} + 2(1 - \cos\varphi) = \frac{4E}{E_{\mathrm{m}}}. \tag{4}$$

Several phase trajectories are shown in figure 1 under the graph of $E_{\mathrm{pot}}(\varphi)$.

If the total energy E of the pendulum is less than the height of the potential barrier in figure 1 ($E < 2mgl = E_{\mathrm{m}}$), the pendulum swings between the extreme deflections φ_{m} and $-\varphi_{\mathrm{m}}$. If the amplitude is small ($\varphi_{\mathrm{m}} \ll \pi/2$), the oscillations are nearly sinusoidal, and the phase trajectory is nearly an ellipse. The greater the total energy E, the greater the divergence of the phase trajectory from an ellipse and the greater the difference of the motion from simple harmonic. At large amplitudes the pendulum spends more time near the extreme (turning) points where its direction of motion reverses. The period of motion grows with the amplitude. If the total energy E of the pendulum is greater

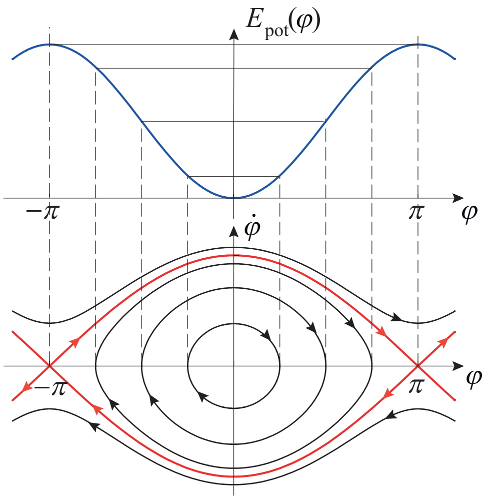

Figure 1. Potential well of the simple pendulum and the phase portrait in the absence of friction. Closed phase trajectories that enclose the origin of the phase plane correspond to oscillations with different amplitudes. Trajectories passing over and below separatrix correspond to counterclockwise and clockwise revolutions respectively.

than the height of the potential barrier ($E > 2mgl = E_{\rm m}$), the pendulum occurs in the inverted position with non-zero angular velocity. This means that it makes full revolutions.

3. The Limiting Motion

The phase trajectory corresponding to a total energy E which is equal to the maximal possible potential energy, namely $E_{\rm pot}(\pi) = E_{\rm m}$, is of special interest. It separates the central region of the phase plane which is occupied by the closed phase trajectories of oscillations from the outer region, occupied by the phase trajectories of rotations. This boundary is called the separatrix. The separatrix divides the phase plane of a conservative pendulum into regions which correspond to different types of motion. The equation of the separatrix follows from equation (3) by setting $E = 2mgl$, or from equation (4) by setting $E = E_{\rm m} = 2ml^2\omega_0^2$:

$$\dot{\varphi} = \pm 2\omega_0 \cos(\varphi/2). \tag{5}$$

When the pendulum with the energy $E = 2mgl$ approaches the inverted position at $\varphi = \pi$ or $\varphi = -\pi$, its velocity tends to zero, becoming zero at $\varphi = \pm\pi$. This state is represented in the phase plane by the saddle points $(\pi, 0)$ and $(-\pi, 0)$ where the upper and lower branches of the separatrix (equation (5)) meet on the φ-axis. Both these points represent the same mechanical state of the system, namely the state in which the pendulum is at rest in the unstable inverted position. The slightest initial displacement of the pendulum from this point to one side or the other results in its swinging with an amplitude which almost equals π, and the slightest initial push causes rotational motion of the pendulum in a full circle. Executing such swinging or rotation, the pendulum spends an extended time in the vicinity of the inverted position.

For the case of motion with the energy $E = E_{\rm m} = 2mgl$ (motion along the separatrix) there exists an analytical solution (in elementary functions) for the angle of deflection $\varphi(t)$ and for the angular velocity $\dot{\varphi}(t)$. Indeed, integration of equation (5) for the positive sign at the initial condition $\varphi(0) = 0$ yields:

$$\varphi(t) = \pi - 4\arctan(e^{-\omega_0 t}). \tag{6}$$

This solution describes a counterclockwise motion beginning at $t = -\infty$ from $\varphi = -\pi$. At $t = 0$ the pendulum passes through the bottom of its circular path, and continues its motion until $t = +\infty$, asymptotically approaching $\varphi = +\pi$. Differentiating $\varphi(t)$ given by equation (6) with respect to time t, we find the following time dependence of the angular velocity $\dot{\varphi}(t)$ for the limiting motion of the pendulum:

$$\dot{\varphi}(t) = \frac{2\omega_0}{\cosh(\omega_0 t)} = \frac{4\omega_0}{e^{\omega_0 t} + e^{-\omega_0 t}}. \tag{7}$$

The graphs of $\varphi(t)$ and $\dot{\varphi}(t)$ for the limiting motion are shown in figure 2. The graph of $\dot{\varphi}(t)$ has the form of an isolated impulse. In equation (7) the time origin $t = 0$ is the instant at which the pendulum passes through the equilibrium position with the angular velocity $\dot{\varphi} = 2\omega_0$. This moment corresponds to the peak value of the impulse. For time $t = \pm T_0/2$ on either side of the peak equation (7) gives the angular velocity of only $\pm 0.17\,\omega_0$. Therefore the period $T_0 = 2\pi/\omega_0$ of small natural oscillations gives an estimate for the duration of the impulse on the velocity graph, that is, for the time needed for the pendulum to execute almost all of its circular path, from the vicinity of the inverted position through the lowest point and back.

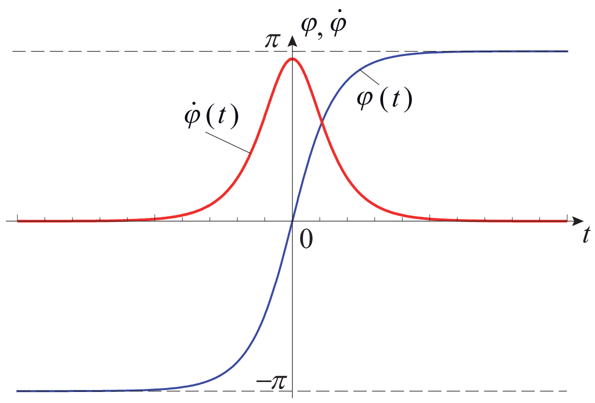

Figure 2. Time dependent graphs of $\varphi(t)$ and $\dot{\varphi}(t)$ for the limiting motion along the separatrix from $\varphi = -\pi$ to $\varphi = \pi$ in the absence of friction. The time origin $t = 0$ corresponds to the moment, at which the pendulum crosses the lower equilibrium position $\varphi = 0$.

4. Oscillations with amplitudes approaching 180°

If the pendulum is released with zero initial velocity near the inverted position (say, at initial angle about 179°), it slowly starts moving toward the down position with a small initial acceleration, because the torque of gravity, being proportional to the sine of deviation from the inverted position, is small. After the pendulum gains some speed, it rapidly makes almost a full circular path through the lower equilibrium position. When the pendulum occurs on the opposite side of the inverted position, its motion gradually slows down as it climbs up along the slope of the potential barrier to its summit. In the absence of friction the pendulum stops when its angular distance to the vertical becomes equal to the initial deviation. From this turning point all the motion repeats in the opposite direction, and after a period the pendulum occurs at the initial point with zero velocity.

A computer simulation of motion of the rigid planar pendulum developed by the author can be found on the web [8]. The simulation program (applet) runs directly in any web browser with Java runtime environment (JRE) installed. To observe the oscillations discussed in this section, we should switch off the viscous friction (using the corresponding check-box on the "parameters" panel), and choose appropriate initial conditions (initial angle about 179°, initial velocity zero). The program allows the user to plot the time dependencies of $\varphi(t)$ and $\dot{\varphi}(t)$, and to draw the phase trajectory simultaneously with the visualization of oscillations.

Graphs of $\varphi(t)$ and $\dot{\varphi}(t)$ for oscillations with amplitudes 179.90° and 179.99° in the absence of friction are shown in figure 3.

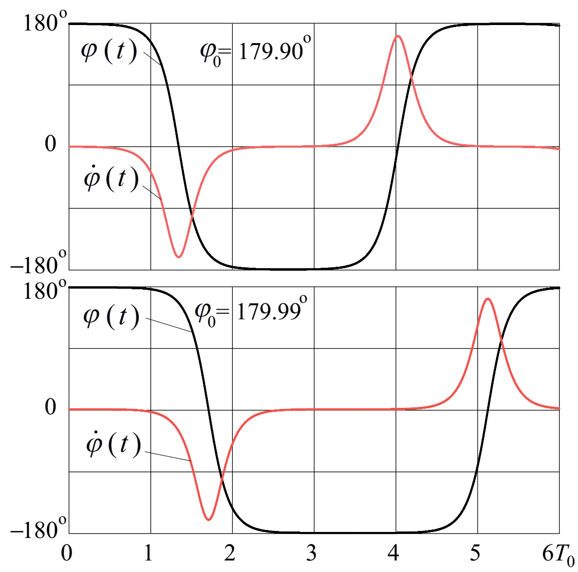

Figure 3. Graphs of $\varphi(t)$ and $\dot{\varphi}(t)$ for oscillations with the amplitude 179.90° and 179.99° in the absence of friction, obtained in the simulation experiment.

Comparing these graphs, we can see that for the most part of the angular excursion from $-\pi$ to π these graphs for amplitudes 179.90° and 179.99° are nearly identical. We guess that for these stages of motion deflection angle $\varphi(t)$ and angular velocity $\dot{\varphi}(t)$ are characterized by almost the same time dependence as for the limiting motion along the separatrix, shown in figure 2. This time dependence is described (in elementary functions) by the simple expression (6). Hence the duration of this stage of oscillation for all these cases of large amplitudes approaching 180° is about T_0 (the period of small oscillations) and can be calculated with high precision with the help of the same expression (6). The duration of the remaining stage, during which the pendulum lingers near the inverted position, depends critically on the amplitude φ_m. This is clearly seen from comparison of the upper and lower panels of figure 3. This duration increases indefinitely as $\varphi_\mathrm{m} \to 180°$. In order to calculate the duration of this stage for certain large amplitudes, we can make use of the linearized differential equation, applicable for small deviations from the inverted position. We will do this on page 8.

The closed phase trajectory of oscillatory motion with a large amplitude φ_m is shown in figure 4. Most part of the phase trajectory almost coincides with the separatrix. The representing point goes around the whole closed curve during one period T of oscillation. Next we consider one quarter of this curve which starts in the phase plane at the initial

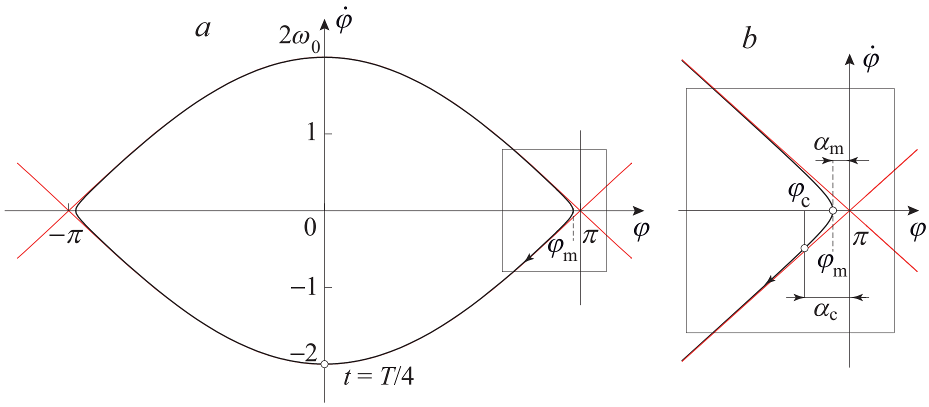

Figure 4. The phase trajectory of oscillatory motion with a large amplitude φ_m (a) and its portion (increased) that corresponds to the motion of the pendulum in the vicinity of the inverted position (b).

point of maximal deflection $\varphi = \varphi_\mathrm{m}$ and initial velocity $\dot{\varphi}(t) = 0$, and ends at the point $\varphi = 0$ (marked as $t = T/4$ in figure 4, a). To calculate this time $t = T/4$, we choose on this curve an arbitrary point $\varphi = \varphi_\mathrm{c}$ not far from the inverted position $\varphi = \pi$ (see figure 4, b), which divides the curve into two parts. The first part between $\varphi = \varphi_\mathrm{m}$ and $\varphi = \varphi_\mathrm{c}$ lies in the vicinity of the inverted position, so that duration t_1 of motion along this part can be calculated with the help of a linearized differential equation of motion (see below). The second part between $\varphi = \varphi_\mathrm{c}$ and $\varphi = 0$ is almost indistinguishable from the separatrix, so that duration t_2 of motion along this part can be immediately expressed with the help of equation (6):

$$\omega_0 t_2 = -\ln\tan\frac{\pi - \varphi_{\rm c}}{4} = -\ln\tan\frac{\alpha_{\rm c}}{4} \approx \ln\frac{4}{\alpha_{\rm c}}. \qquad (8)$$

Here we introduced the notation $\alpha_{\rm c} = \pi - \varphi_{\rm c}$ for the angle that the pendulum makes with the upward vertical line at $\varphi = \varphi_{\rm c}$. When $\varphi_{\rm c}$ is close to π, the angle $\alpha_{\rm c}$ is small, so that in equation (8) we can assume $\tan(\alpha_{\rm c}/4) \approx \alpha_{\rm c}/4$. Therefore $\omega_0 t_2 \approx \ln(4/\alpha_{\rm c})$.

When considering the motion of the pendulum in the vicinity of the inverted position, we find it convenient to define the pendulum position (instead of the angle φ) by the angle α of deflection from the position of unstable equilibrium. This angle equals $\pi - \varphi$, so that $\varphi = \pi - \alpha$. Substituting angular acceleration $\ddot{\varphi} = -\ddot{\alpha}$ and $\sin\varphi = \sin\alpha$ in equation (2), we find the differential equation for the pendulum in terms of α. Since near the inverted position $\alpha \ll 1$, we can replace in this equation $\sin\alpha$ by α. Thus we get the following linear differential equation approximately valid for the pendulum's motion between $\varphi = \varphi_{\rm m}$ and $\varphi = \varphi_{\rm c}$:

$$\ddot{\alpha} - \omega_0^2\alpha = 0. \qquad (9)$$

The general solution to this linear equation can be represented as a superposition of two exponential functions of time t:

$$\alpha(t) = C_1 e^{\omega_0 t} + C_2 e^{-\omega_0 t}. \qquad (10)$$

The initial conditions for the motion from $\varphi = \varphi_{\rm m}$ to $\varphi = \varphi_{\rm c}$ are $\alpha(0) = \alpha_{\rm m}$ and $\dot{\alpha}(0) = 0$. Applying these conditions, we find the constants C_1 and C_2 in equation (10):

$$\alpha(t) = \frac{1}{2}\alpha_{\rm m}(e^{\omega_0 t} + e^{-\omega_0 t}) = \alpha_{\rm m}\cosh\omega_0 t. \qquad (11)$$

To find the duration t_1 of motion from $\varphi = \varphi_{\rm m}$ to $\varphi = \varphi_{\rm c}$ (from $\alpha = \alpha_{\rm m}$ to $\alpha = \alpha_{\rm c}$), we substitute in equation (11) $\alpha(t_1) = \alpha_{\rm c}$:

$$\alpha_{\rm c} = \frac{1}{2}\alpha_{\rm m}(e^{\omega_0 t_1} + e^{-\omega_0 t_1}) \approx \frac{1}{2}\alpha_{\rm m} e^{\omega_0 t_1}. \qquad (12)$$

We have omitted the second term in the right-hand side of equation (12). This is admissible if the arbitrary angle $\alpha_{\rm c}$ (which divides the phase trajectory into two parts) is chosen to be large compared to $\alpha_{\rm m}$. From equation (12) we get for t_1:

$$\omega_0 t_1 = \ln\frac{2\alpha_{\rm c}}{\alpha_{\rm m}}. \qquad (13)$$

The desired period of oscillations T is four times greater than the duration $t_1 + t_2$ of motion from $\varphi = \varphi_{\rm m}$ to the lower equilibrium position $\varphi = 0$. Adding t_1 from equation (13) and t_2 from (8), we finally obtain the following expression for the period of oscillations with large amplitude $\varphi_{\rm m}$ approaching 180°:

$$T = 4(t_1 + t_2) = \frac{4}{\omega_0}(\ln\frac{2\alpha_{\rm c}}{\alpha_{\rm m}} + \ln\frac{4}{\alpha_{\rm c}}) = \frac{2}{\pi}T_0\ln\frac{8}{\alpha_{\rm m}}. \qquad (14)$$

(Here $\alpha_{\rm m} = \pi - \varphi_{\rm m}$.) We note that both t_1 and t_2 depend on the value $\alpha_{\rm c}$ of the angle which we have chosen to divide the trajectory into one part that corresponds to the motion in the vicinity of the inverted position, and the other that almost merges with the separatrix. Nevertheless, this dependence on α_c disappears when we add t_1 and t_2: the final expression (14) for the period is independent of the arbitrarily chosen value of α_c (provided $\alpha_m \ll \alpha_c \ll 1$).

The approximation given by expression (14) is more accurate the closer the amplitude φ_m to 180°. The table below illustrates the precision of this simple expression for oscillations of extremely large amplitudes. The values of T in the middle column are calculated on the basis of exact formula (1); the right column corresponds to the approximate expression (14).

Amplitude φ_m	(α_m)	T/T_0 (exact value)	T/T_0 (approximate)
175.000°	(5.000°)	2.87766	2.87639
177.000°	(3.000°)	3.20211	3.20160
179.000°	(1.000°)	3.90107	3.90099
179.900°	(0.100°)	5.36687	5.36687
179.990°	(0.010°)	6.83274	6.83274
179.999°	(0.001°)	8.29861	8.29861

We note that according to this table one cycle of the pendulum oscillation at large amplitudes covers several periods of small oscillations. As an assignment for students' activity, it would be expedient to suggest them to verify the values cited in the table by direct measurements of the period in a simulation experiment using the software available on the web [8].

The above described oscillations with extremely large amplitudes occur if the energy supplied to the pendulum at initial excitation is slightly less than the height of the potential barrier $E_m = 2mgl$ (see figure 1). If the pendulum is excited from the lower equilibrium position by an initial push, the initial velocity $\dot{\varphi}(0)$ should be a little less than $2\omega_0$. If $\dot{\varphi}(0) > 2\omega_0$, the pendulum makes revolutions in a full circle. If $\dot{\varphi}(0)$ is only slightly greater than $2\omega_0$, it is also expedient to divide the motion of the pendulum into two stages. The stage of motion at crossing the inverted position and in a small vicinity of it can be described with good precision by the linearized equation (9). The remaining almost closed part of the circular path can be approximated, like in the above analysis of oscillations, by the known analytical solution for the limiting motion, equation (6). In this way a simple analytical expression similar to equation (14) for the period of such non-uniform revolutions can be obtained (see [7]).

5. Another derivation of the expression for the period of large oscillations

In the above derivation of expression (14) we have chosen arbitrarily some small angle α_c for dividing the motion into stages described by different analytical time dependencies. Another way is to choose for this conventional boundary of the two stages, instead of the angular position α_c, some arbitrary small angular velocity $\omega_c \ll \omega_0$, which the pendulum gains while moving from the turning point $\alpha_{\rm m}$ at which its angular velocity is zero. To find the duration t_3 of this stage occurring in the vicinity of the inverted position, we can make use of the above obtained solution (11) to the linearized equation (9), according to which

$$\dot\alpha(t) = \frac{1}{2}\alpha_{\rm m}\omega_0(e^{\omega_0 t} + e^{-\omega_0 t}). \qquad (15)$$

Substituting $\dot\alpha(t_3) = \omega_{\rm c}$ in equation (15) and taking into account that $e^{-\omega_0 t_3} \ll e^{\omega_0 t_3}$, we find

$$\omega_0 t_3 = \ln\frac{2\omega_{\rm c}}{\omega_0\alpha_{\rm m}}. \qquad (16)$$

The further motion towards the equilibrium position is almost indistinguishable from the limiting motion. Hence the time dependence of the angular velocity $\dot\alpha(t) = -\dot\varphi(t)$ for this stage can be assumed the same as for the limiting motion, see equation (7). Therefore for calculating the duration t_4 of this stage we can substitute $\dot\varphi(t_4) = \omega_{\rm c}$ in (7) and take into account that $e^{-\omega_0 t_4} \ll e^{\omega_0 t_4}$. This yields

$$\omega_0 t_4 = \ln\frac{4\omega_0}{\omega_{\rm c}}. \qquad (17)$$

Adding t_3 from equation (16) and t_4 from (17), we finally obtain the same simple expression (14) for the period of oscillations with a very large amplitude $\varphi_{\rm m}$:

$$T = 4(t_3 + t_4) = \frac{4}{\omega_0}(\ln\frac{2\omega_{\rm c}}{\omega_0\alpha_{\rm m}} + \ln\frac{4\omega_0}{\omega_{\rm c}}) = \frac{2}{\pi}T_0\ln\frac{8}{\alpha_{\rm m}}. \qquad (18)$$

Again, the arbitrarily chosen angular velocity $\omega_{\rm c}$ ($\omega_{\rm c} \ll \omega_0$) which we have used to divide the motion on different stages vanish from the final expression (18).

6. Concluding remarks

We have considered the old problem of large oscillations of a simple rigid pendulum with amplitudes close to 180° on the basis of an approach in which the cycle of oscillation is divided into several stages. The major part of the almost closed circular path of the pendulum in such an oscillation is approximated with a good precision by the limiting motion, for which there exists an analytic solution in elementary functions. The remaining small parts of the path, occurring in the vicinity of the inverted position, are described on the basis of the linearized equation, which is valid for the new variable $\alpha = \pi - \varphi$. The point that divides the path of the pendulum into stages described by different approximations is chosen to some extent arbitrarily, but this arbitrariness does not influence the final approximate expression for the period of oscillations. The final analytical formula (14) is very simple and gives for the period of large oscillations rather accurate values that coincide with high precision with the values given by the exact expression (1) in terms of the complete elliptic integral of the first kind. More importantly, the approach to the problem described in this letter provides additional physical insight into the dynamics of nonlinear systems.

References

[1] Johannessen K 2010 An approximate solution to the equation of motion for large-angle oscillations of the simple pendulum with initial velocity *Eur. J. Phys.* **31** 511–18

[2] Belendez A, Rodes J J, Hernandez A and Belendez T 2009 Approximation for a large-angle simple pendulum period *Eur. J. Phys.* **30** L25–8

[3] Carvalhaes C G, Suppes P 2008 Approximations for the period of the simple pendulum based on the arithmetic-geometric mean *Am. J. Phys.* **76** 1150–4

[4] Lima F M S 2009 Simple log formulae for the pendulum motion valid for any amplitude *Eur. J. Phys.* **29** 1091–8

[5] Lima F M S, Arun P 2006 An accurate formula for the period of a simple pendulum oscillating beyond the small angle regime *Am. J. Phys.* **74** 892–5

[6] Xin Y Q and Pei D 2009 Comment on Approximation for a large-angle simple pendulum period *Eur. J. Phys.* **30** L79–82

[7] Butikov E 1999 The rigid pendulum – an antique but evergreen physical model *Eur. J. Phys.* **20** 429–41

[8] Butikov E 1996 Free Oscillations and Rotations of a Rigid Pendulum (computer simulation) `http://faculty.ifmo.ru/butikov/Applets/Pendulum.html`